MONASH University

MONASH BUSINESS SCHOOL

ETF3231/5231

Business forecasting

Week 9: ARIMA models

https://bf.numbat.space/

SUMMARY

$$\phi(B)(1-B)^d y_t = c + \theta(B)\varepsilon_t$$

$(p, q) \to$ short-run
$(c, d) \to$ long-run

constant

$c = 0$	$d = 0$	$\hat{y}_{T+\infty} \to 0$	ARMA stat around	$E(y_t) = 0$
$c \neq 0$	$d = 0$	$\hat{y}_{T+\infty} \to E(y_t)$	" " "	$E(y_t) \neq 0$
$c = 0$	$d = 1$	$\hat{y}_{T+\infty} \to$ const	RW + ARMA	• diff of RW is stat and the ARMA part will converge to const.

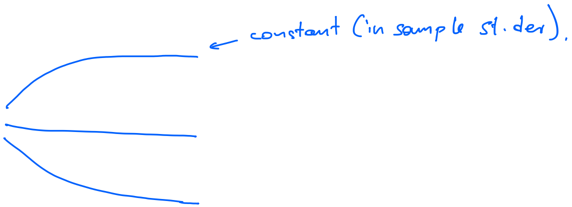
linear trend

$c \neq 0$	$d = 1$	$\hat{y}_{T+\infty} \to t$	RW + drift + ARMA
$c = 0$	$d = 2$		Two unit roots

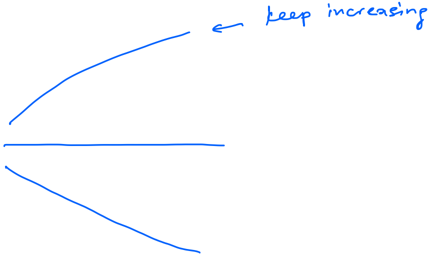
quadratic trend

$c \neq 0$ $d = 2$ Do IT AT YOUR OWN RISK.

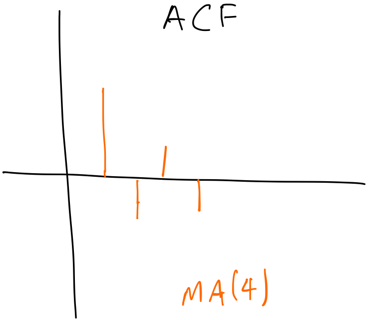
$d = 0$

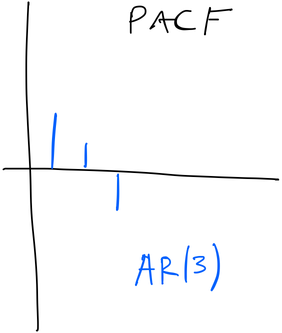


$d = 1$



EXTRA JUST TO RECAP *





* from theory / reading ACF / PACF we can tell either AR or MA orders

Outline

1 ARIMA modelling in R

2 Forecasting

3 Seasonal ARIMA models

4 ARIMA vs ETS

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Modelling procedure with `ARIMA()`

1. Plot the data. Identify any unusual observations. Always
2. If necessary, transform the data (using a Box-Cox transformation) to stabilize the variance. remove heteroscedasticity, make variance stationary
3. If the data are non-stationary: take first differences of the data until the data are stationary. make stationary in the mean
4. Examine the ACF/PACF: Is an AR(p) or MA(q) model appropriate? - select model(s). - Most likely more than one
5. Try your chosen model(s), and use the AICc to search for a better model.
6. Check the residuals from your chosen model by plotting the ACF of the residuals, and doing a portmanteau test of the residuals. If they do not look like white noise, try a modified model.
7. Once the residuals look like white noise, calculate forecasts.

ARIMA() (steps 3–5)

Talk about JA4. Remember models are DGP approximations. You can only do your best (95% v 70% coverage)

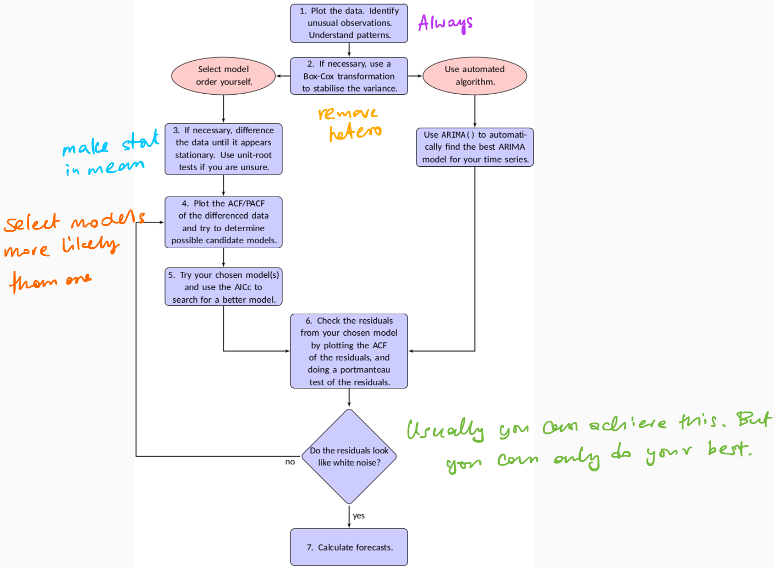
Automatic modelling procedure with `ARIMA()`

1. Plot the data. Identify any unusual observations.
2. If necessary, transform the data (using a Box-Cox transformation) to stabilize the variance.

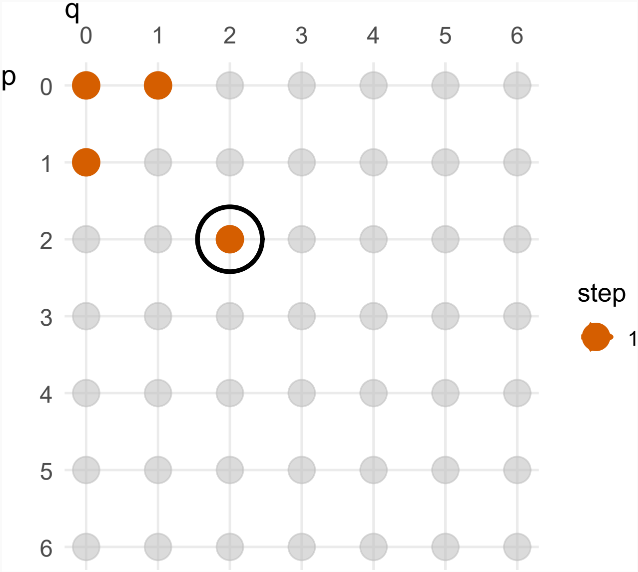
3. Use `ARIMA()` to automatically select a model.

6. Check the residuals from your chosen model by plotting the ACF of the residuals, and doing a portmanteau test of the residuals. If they do not look like white noise, try a modified model.
7. Once the residuals look like white noise, calculate forecasts.

Modelling procedure



How does ARIMA() work?



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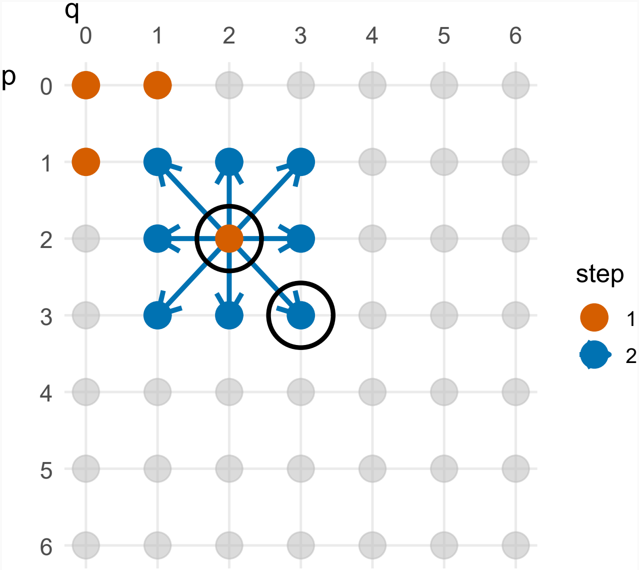

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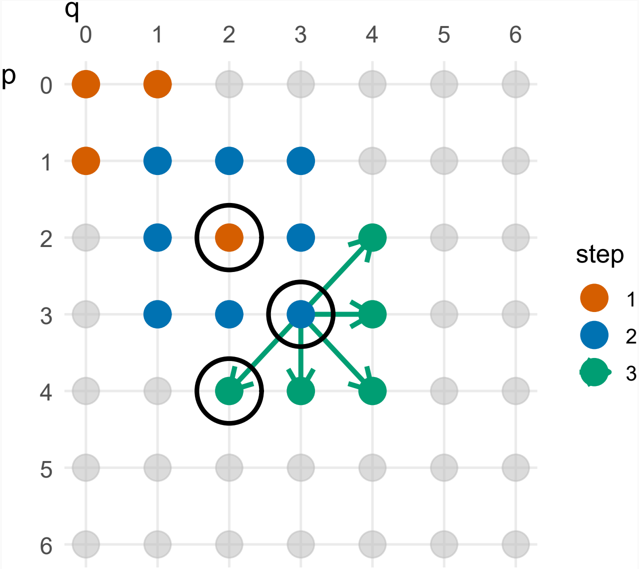

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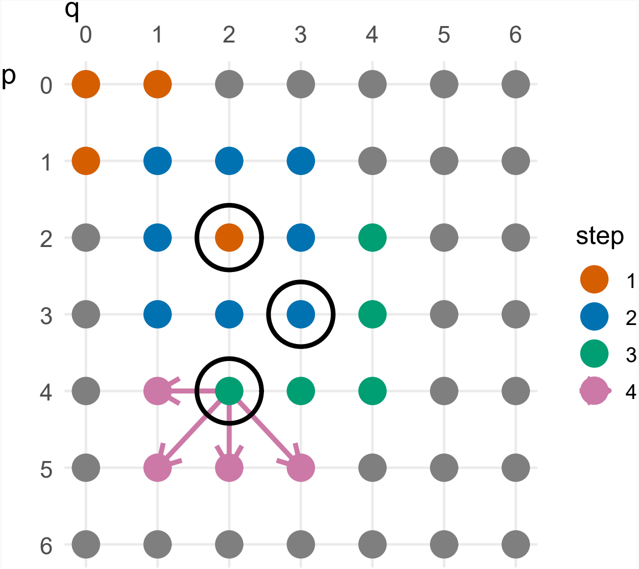

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Point forecasts

1. Rearrange ARIMA equation so y_t is on LHS.
2. Rewrite equation by replacing t by $T + h$.
3. On RHS, replace future observations by their forecasts, future errors by zero, and past errors by corresponding residuals.

Start with $h = 1$. Repeat for $h = 2, 3, \dots$.

Point forecasts

ARIMA(3,1,1) forecasts: Step 1

$$(1 - \hat{\phi}_1 B - \hat{\phi}_2 B^2 - \hat{\phi}_3 B^3)(1 - B) y_t = (1 + \hat{\theta}_1 B)\varepsilon_t,$$

Point forecasts

ARIMA(3,1,1) forecasts: Step 1

$$(1 - \hat{\phi}_1 B - \hat{\phi}_2 B^2 - \hat{\phi}_3 B^3)(1 - B) y_t = (1 + \hat{\theta}_1 B)\varepsilon_t,$$

$$\left[1 - (1 + \hat{\phi}_1)B + (\hat{\phi}_1 - \hat{\phi}_2)B^2 + (\hat{\phi}_2 - \hat{\phi}_3)B^3 + \hat{\phi}_3 B^4\right] y_t = (1 + \hat{\theta}_1 B)\varepsilon_t,$$

$$y_t - (1 + \hat{\phi}_1)y_{t-1} + (\hat{\phi}_1 - \hat{\phi}_2)y_{t-2} + (\hat{\phi}_2 - \hat{\phi}_3)y_{t-3} + \hat{\phi}_3 y_{t-4} = \varepsilon_t + \hat{\theta}_1 \varepsilon_{t-1}.$$

$$y_t = (1 + \hat{\phi}_1)y_{t-1} - (\hat{\phi}_1 - \hat{\phi}_2)y_{t-2} - (\hat{\phi}_2 - \hat{\phi}_3)y_{t-3} - \hat{\phi}_3 y_{t-4} + \varepsilon_t + \hat{\theta}_1 \varepsilon_{t-1}.$$

Point forecasts (h=1)

$$y_t = (1+\hat\phi_1)y_{t-1} - (\hat\phi_1 - \hat\phi_2)y_{t-2} - (\hat\phi_2 - \hat\phi_3)y_{t-3} - \hat\phi_3 y_{t-4} + \varepsilon_t + \hat\theta_1\varepsilon_{t-1}.$$

Point forecasts (h=1)

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ARIMA(3,1,1) forecasts: Step 2

$$y_{T+1} = (1 + \hat{\phi}_1)y_T - (\hat{\phi}_1 - \hat{\phi}_2)y_{T-1} - (\hat{\phi}_2 - \hat{\phi}_3)y_{T-2} - \hat{\phi}_3 y_{T-3} + \varepsilon_{T+1} + \hat{\theta}_1\varepsilon_T.$$

Point forecasts (h=1)

$$y_t = (1 + \hat{\phi}_1)y_{t-1} - (\hat{\phi}_1 - \hat{\phi}_2)y_{t-2} - (\hat{\phi}_2 - \hat{\phi}_3)y_{t-3} - \hat{\phi}_3 y_{t-4} + \varepsilon_t + \hat{\theta}_1\varepsilon_{t-1}.$$

ARIMA(3,1,1) forecasts: Step 2

$$y_{T+1} = (1 + \hat{\phi}_1)y_T - (\hat{\phi}_1 - \hat{\phi}_2)y_{T-1} - (\hat{\phi}_2 - \hat{\phi}_3)y_{T-2} - \hat{\phi}_3 y_{T-3} + \varepsilon_{T+1} + \hat{\theta}_1\varepsilon_T.$$

$E(\varepsilon_{T|T}) = e_T$

$var(\varepsilon_{T|T}) = 0$

$$E(y_{T+1/T}) = (1 + \hat{\phi}_1)y_T - (\hat{\phi}_1 - \hat{\phi}_2)y_{T-1} - (\hat{\phi}_2 - \hat{\phi}_3)y_{T-2} - \hat{\phi}_3 y_{T-3} + E(\varepsilon_{T+1/T}) + \hat{\theta}_1 E(\varepsilon_{T/T}).$$

$E(\varepsilon_{T+1|T}) = 0$

$var(\varepsilon_{T+1|T}) = \sigma^2$

Point forecasts (h=1)

$$y_t = (1 + \hat{\phi}_1)y_{t-1} - (\hat{\phi}_1 - \hat{\phi}_2)y_{t-2} - (\hat{\phi}_2 - \hat{\phi}_3)y_{t-3} - \hat{\phi}_3 y_{t-4} + \varepsilon_t + \hat{\theta}_1\varepsilon_{t-1}.$$

ARIMA(3,1,1) forecasts: Step 2

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0 e_T

$E(\varepsilon_{T|T}) = e_T$

$var(\varepsilon_{T|T}) = 0$

$E(\varepsilon_{T+1|T}) = 0$

$var(\varepsilon_{T+1|T}) = \sigma^2$

Point forecasts (h=1)

$$y_t = (1 + \hat{\phi}_1)y_{t-1} - (\hat{\phi}_1 - \hat{\phi}_2)y_{t-2} - (\hat{\phi}_2 - \hat{\phi}_3)y_{t-3} - \hat{\phi}_3 y_{t-4} + \varepsilon_t + \hat{\theta}_1\varepsilon_{t-1}.$$

ARIMA(3,1,1) forecasts: Step 2

$$y_{T+1} = (1 + \hat{\phi}_1)y_T - (\hat{\phi}_1 - \hat{\phi}_2)y_{T-1} - (\hat{\phi}_2 - \hat{\phi}_3)y_{T-2} - \hat{\phi}_3 y_{T-3} + \varepsilon_{T+1} + \hat{\theta}_1\varepsilon_T.$$

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ARIMA(3,1,1) forecasts: Step 3

$$\hat{y}_{T+1|T} = (1 + \hat{\phi}_1)y_T - (\hat{\phi}_1 - \hat{\phi}_2)y_{T-1} - (\hat{\phi}_2 - \hat{\phi}_3)y_{T-2} - \hat{\phi}_3 y_{T-3} + \hat{\theta}_1 e_T.$$

Point forecasts (h=2)

$$y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} - \phi_3 y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.$$

Point forecasts (h=2)

$$y_t = (1+\phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} - \phi_3 y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.$$

ARIMA(3,1,1) forecasts: Step 2

$$E(y_{T+2/T}) = (1+\hat{\phi}_1)E(y_{T+1/T}) - (\hat{\phi}_1 - \hat{\phi}_2)y_T - (\hat{\phi}_2 - \hat{\phi}_3)y_{T-1} - \hat{\phi}_3 y_{T-2} + E(\varepsilon_{T+2/T}) + \hat{\theta}_1 E(\varepsilon_{T+1/T}).$$

Point forecasts (h=2)

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$\hat{y}_{T+1|T}$ 0 0

Point forecasts (h=2)

$$y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} - \phi_3 y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.$$

ARIMA(3,1,1) forecasts: Step 2

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ARIMA(3,1,1) forecasts: Step 3

$$\hat{y}_{T+2|T} = (1 + \phi_1)\hat{y}_{T+1|T} - (\phi_1 - \phi_2)y_T - (\phi_2 - \phi_3)y_{T-1} - \phi_3 y_{T-2}.$$

Prediction intervals

95% prediction interval

$$\hat{y}_{T+h|T} \pm 1.96\sqrt{v_{T+h|T}}$$

where $v_{T+h|T}$ is estimated forecast variance.

Prediction intervals

95% prediction interval

$$\hat{y}_{T+h|T} \pm 1.96\sqrt{v_{T+h|T}}$$

where $v_{T+h|T}$ is estimated forecast variance.

true for any time series model

- $v_{T+1|T} = \hat{\sigma}^2$ for all ARIMA models regardless of parameters and orders.
- Multi-step prediction intervals for ARIMA(0,0,q):

$$y_t = \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}.$$

very easy to do (P.T.O.)

$$v_{T|T+h} = \hat{\sigma}^2 \left[1 + \sum_{i=1}^{h-1} \theta_i^2 \right], \qquad \text{for } h = 2, 3, \ldots.$$

* AR(1) -> MA(∞)
* more complex beyond our scope

$MA(q)$ $\quad y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \dots + \theta_q \varepsilon_{t-q}$

- $var(\theta \varepsilon_t) = \theta^2 var(\varepsilon_t) = \theta^2 \sigma^2$ for $t > T$

$$y_{T+1} = c + \varepsilon_{T+1} + \theta_1 \varepsilon_T + \theta_2 \varepsilon_{T-1} + \dots + \theta_q \varepsilon_{T-q+1}$$

$$var(y_{T+1|T}) = var(c + \varepsilon_{T+1} + \theta_1 \varepsilon_T + \theta_2 \varepsilon_{T-1} + \dots + \theta_q \varepsilon_{T-q+1})$$

$$= var(\varepsilon_{T+1|T}) = \sigma^2$$

(Everything else has been observed hence $var(\varepsilon_t) = 0$ $t \leq T$)

$$var(y_{T+2|T}) = var(c + \varepsilon_{T+2} + \theta_1 \varepsilon_{T+1} + \theta_2 \varepsilon_T + \theta_3 \varepsilon_{T-1} + \dots + \theta_q \varepsilon_{T-q+2})$$

$$= var(\varepsilon_{T+2|T}) + \theta_1^2 var(\varepsilon_{T+1|T})$$

$$= \sigma^2 + \theta_1^2 \sigma^2 = (1 + \theta_1^2)\sigma^2$$

and so on .... $(1 + \theta_1^2 + \theta_2^2)\sigma^2 \dots$

Prediction intervals

* remember the role of d ξ

- Prediction intervals increase in size with forecast horizon.
- Prediction intervals can be difficult to calculate by hand
- Calculations assume residuals are uncorrelated and normally distributed.

Prediction intervals

* remember the role of d ?

- Prediction intervals increase in size with forecast horizon.
- Prediction intervals can be difficult to calculate by hand
- Calculations assume residuals are uncorrelated and normally distributed.
- Prediction intervals tend to be too narrow.
  - the uncertainty in the parameter estimates has not been accounted for.
  - the ARIMA model assumes historical patterns will not change during the forecast period. → beware of structural changes (eg. GFC, Covid, etc.).
  - the ARIMA model assumes uncorrelated future errors.

+ correct model to start with

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1 ARIMA modelling in R

2 Forecasting

3 Seasonal ARIMA models — not much more to learn.
- Basically implement you knowledge to now include the seasonal frequency.

4 ARIMA vs ETS

Seasonal ARIMA models

ARIMA	$\underbrace{(p, d, q)}$	$\underbrace{(P, D, Q)_m}$
	$\uparrow$ Non-seasonal part of the model	$\uparrow$ Seasonal part of of the model

where m = number of observations per year.

- monthly
- quarterly

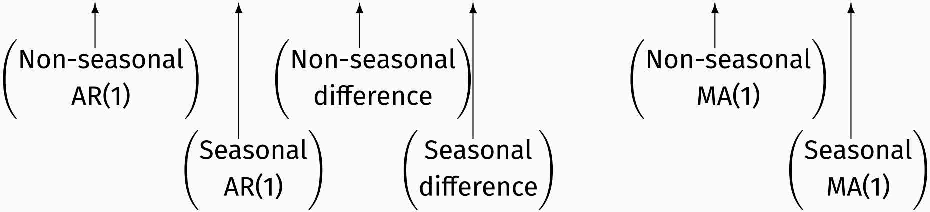
• annual seas. a problem for
- weekly
- daily

m = number of observations per week. - daily

Seasonal ARIMA models

E.g., ARIMA(1, 1, 1)(1, 1, 1)$_4$ model (without constant)

$$(1 - \phi_1 B)(1 - \Phi_1 B^4)(1 - B)(1 - B^4) y_t = (1 + \theta_1 B)(1 + \Theta_1 B^4)\varepsilon_t.$$



Seasonal ARIMA models

E.g., ARIMA$(1,1,1)(1,1,1)_4$ model (without constant)

$$(1-\phi_1 B)(1-\Phi_1 B^4)(1-B)(1-B^4)y_t = (1+\theta_1 B)(1+\Theta_1 B^4)\varepsilon_t.$$

All the factors can be multiplied out and the general model written as follows:

$$\begin{aligned} y_t = {} & (1+\phi_1)y_{t-1} - \phi_1 y_{t-2} + (1+\Phi_1)y_{t-4} \\ & - (1+\phi_1+\Phi_1+\phi_1\Phi_1)y_{t-5} + (\phi_1+\phi_1\Phi_1)y_{t-6} \\ & - \Phi_1 y_{t-8} + (\Phi_1+\phi_1\Phi_1)y_{t-9} - \phi_1\Phi_1 y_{t-10} \\ & + \varepsilon_t + \theta_1\varepsilon_{t-1} + \Theta_1\varepsilon_{t-4} + \theta_1\Theta_1\varepsilon_{t-5}. \end{aligned}$$

* very messy
* unintuitive

Seasonal ARIMA models

The seasonal part of an AR or MA model will be seen in the seasonal lags of the PACF and ACF.

ARIMA(0,0,0)(0,0,1)$_{12}$ will show:

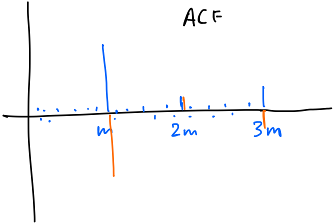
- a spike at lag 12 in the ACF but no other significant spikes.
- The PACF will show exponential decay in the seasonal lags; that is, at lags 12, 24, 36, ...

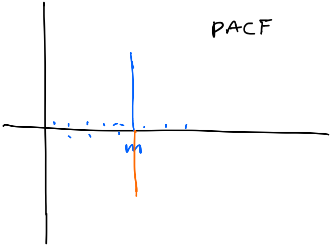
* Not more than 1 or 2 (max)

ARIMA(0,0,0)(1,0,0)$_{12}$ will show:

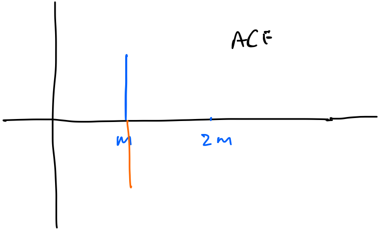
- exponential decay in the seasonal lags of the ACF
- a single significant spike at lag 12 in the PACF.

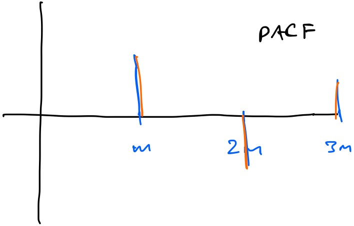
Seasonal AR(1)





Seasonal MA(1)





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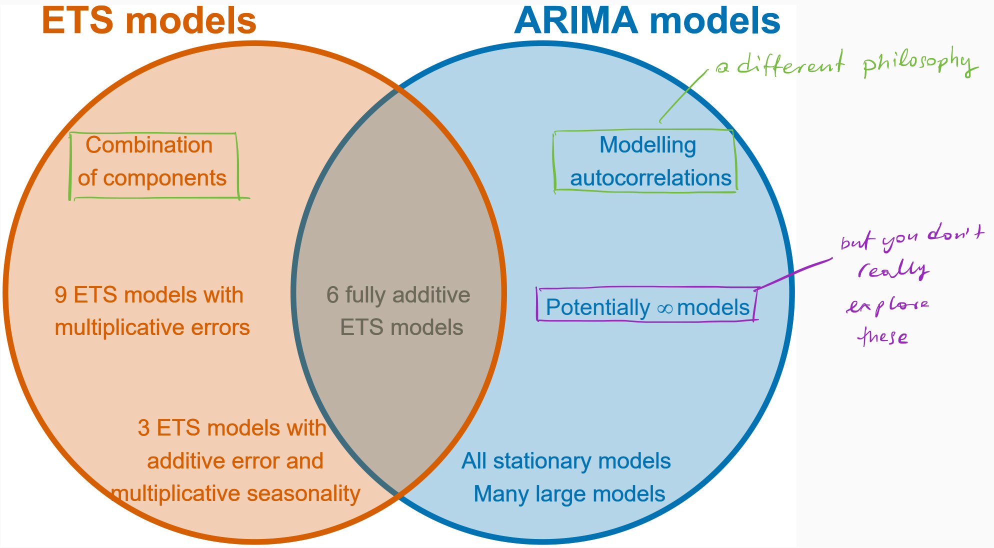
ARIMA vs ETS

- Myth that ARIMA models are more general than exponential smoothing.
- Linear exponential smoothing models all special cases of ARIMA models.
- Non-linear exponential smoothing models have no equivalent ARIMA counterparts.
- Many ARIMA models have no exponential smoothing counterparts.
- ETS models are all non-stationary. Models with seasonality or non-damped trend (or both) have two unit roots; all other models have one unit root.

ARIMA vs ETS



ETS models
ARIMA models
a different philosophy
Combination of components
Modelling autocorrelations
9 ETS models with multiplicative errors
6 fully additive ETS models
Potentially ∞ models
but you don't really explore these
3 ETS models with additive error and multiplicative seasonality
All stationary models
Many large models

Equivalences

ETS model	ARIMA model	Parameters
ETS(A,N,N)	ARIMA(0,1,1)	$\theta_1 = \alpha - 1$
ETS(A,A,N)	ARIMA(0,2,2)	$\theta_1 = \alpha + \beta - 2$ $\theta_2 = 1 - \alpha$
ETS(A,A$_d$,N)	ARIMA(1,1,2)	$\phi_1 = \phi$ $\theta_1 = \alpha + \phi\beta - 1 - \phi$ $\theta_2 = (1 - \alpha)\phi$
ETS(A,N,A)	ARIMA$(0,0,m)(0,1,0)_m$	
ETS(A,A,A)	ARIMA$(0,1,m+1)(0,1,0)_m$	
ETS(A,A$_d$,A)	ARIMA$(1,0,m+1)(0,1,0)_m$	

very easy to show (P.T.O)

These are more complex

ETS (A, N, N) as ARIMA (0, 1, 1) $\theta_1 = \alpha - 1$

$y_t = l_{t-1} + \varepsilon_t$

$l_t = l_{t-1} + \alpha \varepsilon_t$

Diff obs equation $y_t - y_{t-1} = l_{t-1} - l_{t-2} + \varepsilon_t - \varepsilon_{t-1}$

from level equation $l_{t-1} = l_{t-2} + \alpha \varepsilon_{t-1} \Rightarrow l_{t-1} - l_{t-2} = \alpha \varepsilon_{t-1}$

$\Rightarrow \quad y_t - y_{t-1} = \alpha \varepsilon_{t-1} + \varepsilon_t - \varepsilon_{t-1}$

$\Rightarrow \quad y_t = y_{t-1} + \varepsilon_t + (\alpha - 1) \varepsilon_{t-1}$

$\Rightarrow \quad y_t = y_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1}$ where $\theta_1 = \alpha - 1$